TEACHING MATERIAL ON

MATHEMATICS

SCHOOL OF SCIENCE

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Quadratic Programming:

Intro.

We studied LPP. and simplex method (efficient method)

Unlike the LPP case, no such general algorithm exists for solving all NLPP. However for problems with certain suitable structures, efficient algorithms have been developed. Also, it is often possible to convert the given non-linear programming problem into one in which these structures becomes visible.

The general mathematical Programming Problem (GMPP) can be defined as the problem of finding $\bar{x} \in R^n$ so that to

Optimize the obj. fn

(a) $Z = f(\bar{x})$

s.t. Constraints (b) $g_i(x) \, (\leq, = \text{ or } \geq) \, b_i$, $i = 1, \ldots m$

and (c) $\bar{x} \geq 0$,

Where $f(\bar{x})$ & $g_i(\bar{x})$ are the real valued fn of $\bar{x}$ for $i = 1, 2 \cdot\cdot m$ & b_i's are Constants.

It may be observed that the above GMPP reduces to general NLPP (GNLPP) if (a) either $f(\bar{x})$ or $g_i(\bar{x})$ for some or all $i=1,\ldots m$

or $f(\bar{x})$ only

or $g(x)$ only for some or all $i=1,2 \ldots m$

are non-linear in $\bar{x}$. Further, there functions are assumed to be continuously differentiable.

Optimal soln to a NLPP can be found anywhere on the boundary of the feasible region unlike LPP. but there is no such, as yet, any simplex method developed so far to find the optimal soln ~~of a~~ to a NLPP.

A well known quadratic programming model, dealing with the problems of selecting an investment portfolio that will yield a given expected total return with a minimum variance was developed by 'Markowitz'. The problem referred to as the portfolio selection model assumes that the investor wishes to maximize his anticipated returns ~~of~~ while he considers variance of return as undesirable.

Wolf e's Method · (Simplex Method)

Let the quadratic programming problem be 78

$$\text{Max } z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k.$$

S.t. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $x_j \geq 0$ $(i=1,\dots m, j=1,\dots n)$

where $c_{jk} = c_{kj}$ $\forall$ j & k, $b_i \geq 0$ $\forall$ $i=1,\dots m$

Also we assume that the quadratic form $\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$ be negative semi definite.

Then, the wolfe's iterative procedure may be obtained in the following ways.

Step 1: First, convert inequality constraints into equations, by introducing slack variables q_i^2 in the ith constraint $(i=1,\dots m)$ and the slack variables r_j^2 in the jth non-negativity constraint $(j=1,\dots n)$

Step 2 Then construct the Lagrangian fn

$$L(\bar{x}, \bar{q}, \bar{r}, \lambda, \mu) = f(\bar{x}) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2\right]$$
$$- \sum_{j=1}^{n} \mu_j \left[-x_j + r_j^2\right]$$

The soln thus obtained by [illegible] will be optimum soln for app.

where $x = (x_1, x_2 \ldots x_n)'$, $q = (q_1^2, \ldots, q_m^2)'$

$r = (r_1^2, \ldots; r_n^2)$ & $\lambda = (\lambda_1, \ldots \lambda_m)$ (15)

$\mu = (\mu_1, \mu_2 \ldots \mu_n)$.

Differentiating the above fn 'L' partially w.r.t the components x, q, r, λ, μ & equating the first order partial derivatives to zero, we derive Kuhn-Tucker Conditions from the resulting equations.

3. Wolfe (1959) suggested to introduce the non-negative artificial variables v_j, $j = 1, 2 \ldots n$ in the Kuhn-Tucker Conditions

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0$$

for $j = 1, 2 \ldots n$ and construct an objective fn

$$Z_v = v_1 + v_2 + \cdots + v_n$$

Step-4, In this step, obtain the initial basic feasible solution to the following linear programming problem

$$\text{Min } Z_v = v_1 + v_2 + \ldots + v_n.$$

S.t.

the the Constraint

$$\sum_{k=1}^{n} q_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + v_j = -c_j.$$

$(j=1, --n)$

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^2 = b_i \quad (i=1 \cdot\cdot m),$$

$$v_j, \lambda_i, \mu_j, x_j \geq 0 \quad (i=1, --m, \; j=1, --n)$$

and satisfying the complementary Slackness conditions:

$$\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} \lambda_i s_i = 0,$$

(where $s_i = q_i^2$);

$$A_i s_i = 0 \text{ \& } \mu_j x_j = 0 \quad (\text{for } i=1-m, \; j=1-n)$$

Step 5: Now, apply ~~two~~-phase ~~metho~~ simplex method in the usual manner to find an optimal soln to the L.P.P constructed in step 4. The soln. must satisfy the above complementary Slackness condition.

Step-6 The optimum soln thus obtained in step 5 gives the optimum soln of given QPP.

Wolfe's modified simplex method :—

Let the quadratic programming problem be

Maximize $z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$

Subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \le b_i$, $x_j \ge 0$; $(i = 1 \dots m, j = 1 \dots n)$

where $c_{jk} = c_{kj}$ $\forall j$ and k, $b_i \ge 0$ for all $i = 1, 2 \cdots m$.

we also assume that the quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$ be negative - semi definite.

Soln. **Step-1** First convert the inequality constraints into equations by introducing slack-variables q_i^2 in the ith constraints $(i=1 \cdots m)$ and the slack variable r_j^2 in the j^{th} non-negativity constraints $(j=1, \cdots n)$.

Step-2 Then we construct the Lagrangian function

$$L(\bar{x}, \bar{q}, \bar{r}, \bar{\lambda}, \bar{\mu}) = f(\bar{x}) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 \right] - \sum_{j=1}^{n} \mu_j \left[-x_j + r_j^2 \right]$$

when $\bar{x} = (x_1, \cdots x_n)$, $\bar{q} = (q_1 \cdots q_m)$, $\bar{r} = (r_1^2, r_2^2 \cdots r_n^2)$, $\bar{\lambda} = (\lambda_1 \cdots \lambda_m)$, $\bar{\mu} = (\mu_1 \cdots \mu_n)$

differentiating the above function 'L' partially w.r.t $x, \bar{q}, r, \lambda, \mu$, and equating the 1st order p.d -derivative to zero, we derive Kuhn-Tucker conditions from the resulting conditions.

Step-3. Wolfe (1959) suggested to introduce the non-negative artificial variable v_j ; $j = 1, \cdots n$ in the Kuhn Tucker condition.

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0 \quad j = 1, \cdots n.$$

and to construct an objective fn $Z_v = v_1 + v_2 + \cdots + v_n$.

Step-4 In this step we obtain the initial basic solution to the following linear programming problem.

Min $Z_v = v_1 + v_2 + \cdots + v_n$, subject to the constraints :-

$$\sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + v_j = -c_j \quad (j = 1, \cdots n)$$

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^2 = b_i \quad (i = 1, \cdots m),$$

$$v_j, \lambda_j, \mu_j, x_j \ge 0 \quad (i = 1, \cdots m ; j = 1, \cdots n)$$

and satisfying the complementary slackness conditions

$$\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} \lambda_i s_i = 0, \text{ (where } s_i = q_i^2\text{)}, \lambda_i s_i = 0 \text{ \& } \mu_j x_j \text{ (for } i=1, \cdots m, j\text{)}$$

Step-5:- Now apply two phase simplex method to find optimum soln to this LPP. The soln must satisfy the above complementary slackness condition, which will be optimum soln for QPP.//

Example-1 Apply Wolfe's method for solving the quadratic programming problem:— $\boxed{158}$

$$\text{Max } z_x = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2, \text{ subject to}$$

$$x_1 + 2x_2 \le 2, \text{ and } x_1, x_2 \geq 0.$$

Soln:-

Step-1 First we write all the constraint inequalities with $\le$ sign as follows:- $x_1 + 2x_2 \le 2$, $-x_1 \le 0$, $-x_2 \le 0$.

Step-2. Now, introducing the slack variables q_1^2, r_1^2, r_2^2, our problems becomes of the form:—

$$\text{Max } z_x = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

$$\text{Subject to } \begin{array}{rl} x_1 + 2x_2 + q_1^2 & = 2 \\ -x_1 + r_1^2 & = 0 \\ -x_2 + r_2^2 & = 0 \end{array}$$

Step-3. Here to obtain the Kuhn-Tucker conditions, we construct the Lagrange function

$$L(x_1, x_2, q_1, r_1, r_2, \lambda_1, \mu_1, \mu_2)$$

$$= (4x_1 + 6x_2 - 2x_1 - 2x_1x_2 - 2x_2^2) - \lambda_1(x_1 + 2x_2 + q_1^2 - 2) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

The necessary and sufficient conditions are:-

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

Defining $s_i = q_1^2$, we have $\lambda_i s_i = 0$, $\mu_1 x_1 = 0$, $\mu_2 x_2 = 0$.

Also, $x_1 + 2x_2 + s_1 = 2$ and finally $x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2$

Step-4. To construct the modified linear programing problem we will introduce the artificial variables v_1 and v_2 the modified linear programming problem becomes:-

$$\text{Max } z_v = -v_1 - v_2 \quad \text{Subject to}$$

$$\begin{array}{l} 4x_1 + 2x_2 + \lambda_1 - \mu_1 + v_1 = 4 \\ 2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + v_2 = 6 \\ x_1 + 2x_2 + s_1 = 2 \end{array}$$

Where all variables are non-negative and $\mu_1 x_1 = 0$, $\mu_2 x_2 = 0$, $\lambda_1 s_1 = 0$. Now all these constraints equations can be written in matrix form as follows:—

Assignment Problem

Introduction:- It is a special type of linear programming problem in which the objective is to find the optimum allocation of a number of tasks (Jobs) to an equal number of facilities (persons).

Here we have a general assumption that each person can perform each Job but with varying degree of efficiency. For example a departmental head may have four persons available for assignment and four jobs to fill. Then this will be his interest to find out the best assignment in the interest of his department.

Matrix form (standard form) of Assignment problem:-

The assignment problem can be stated in the form of $n \times n$ matrix $[c_{ij}]$ called the cost or effective matrix and means that it is the cost of assigning ith facility (person) to the j-th job, also called as effectiveness matrix.

: Effectiveness Matrix :

Jobs

Persons (facilities)

$$\begin{array}{c|cccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & j & \cdots & n \\
\hline
1 & c_{11} & c_{12} & c_{13} & \cdots & & & & c_{1j} & \cdots & c_{1n} \\
2 & c_{21} & c_{22} & c_{23} & \cdots & & & & c_{2j} & & c_{2n} \\
3 & \cdots & & & & & & & & & \\
\vdots & & & & & & & & & & \\
i & c_{i1} & c_{i2} & c_{i3} & \cdots & & & & c_{ij} & \cdots & c_{in} \\
n & c_{n1} & c_{n2} & c_{n3} & \cdots & & & & c_{nj} & \cdots & c_{nn}
\end{array}$$

②

Mathematical formulation of an Assignment Problem :—

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Defn Minimize the total cost $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$

where, $x_{ij} = \begin{cases} 1 \text{ if i-th person is assigned to the j-th job.} \\ 0, \text{ if i-th person is not assigned to the j-th job.} \end{cases}$

Subject to the conditions that

(i) $\sum_{i=1}^{n} x_{ij} = 1$, $j = 1, 2, \cdots\cdots n$; means only job is done by the i-th person, $i = 1, 2, \cdots,$

(ii) $\sum_{j=1}^{n} x_{ij} = 1$, $i = 1, 2, 3 \cdots, n$; means that only one-person should be assigne to the j-th job, $j = 1, 2, \cdots n$.

Important Theorems:—

Theorem-1 If in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix $[c_{ij}]$, then an assignment which minimizes the total cost for one matrix, also minimizes the total cost for the other matrix.

Mathematically,

If $x_{ij} = X_{ij}$, minimizes $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ over all x_{ij} such that $\sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}$ and $x_{ij} \geq 0$

then $x_{ij} = X_{ij}$ also minimizes $Z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x$

where $c'_{ij} = c_{ij} \pm a_i \pm b_j$, a_i, b_j are constants, $i = 1, 2 \cdots n$; $j = 1, 2, \cdots, n$.

Theorem2: If all $c_{ij} \geq 0$ and $\exists$ a solution $x_{ij} = X_{ij}$ s.t $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 0$ then this solution is an optional solution (i.e, this solution minimizes Z).

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Assignment Problem

Assignment Problem is a particular or special type of Transportation problem in which a number of operations are to be assigned to an equal number of operators.

Hungerian Method (Reduced Matrix Method):-

Step-1 Subtract the minimum elt of each row in the Cost matrix $[C_{ij}]$ from every element of the corresponding row. (called row-operations)

Step-2 Do column operations as said in row operation in step-1. Here at this stage we will get row & column reduced matrix.

Step-3

(a) Now we examine rows successively until a row having exactly one zero is found. Mark (□) at this zero, as an assignment made there. Mark (X) at all other zeros in the column (in which we have marked □) to indicate that they cannot be used to make other assignments. We proceed in this way until the last row is examined.

(b) We adopt the same rule for examining the columns also as done above in case of rows.

(c) We continue these operations (a) and (b) successively until we reach to any of the two situations:—

(i) all the rows are marked □ or ☒ or

(ii) the remaining unmarked zeros lies at least two in each row and column.

For case (i) we have a maximal assignment and still to improve.

(ii) still we have some zeros.

Now there are two possibilities:—

(1) If we have got an assignment in every row and every column i.e. total No. of □ = n (the order

If such condition is reached we say that we have got the complete optimal assignment.

(ii) If above (i) condition [no. of $\square \neq n$] then we will have to modify the cost (effectiveness) matrix by adding or subtracting to create some more zeros in it. For this we proceed as in step 4 below.

Step-4: We draw minimum number of horizontal and vertical lines necessary to cover all zeros at least once. For this the following is adopted:—

(i) Mark (✓) all rows for which assignment have not been made.
(ii) Mark (✓) column which have zero in marked rows.
(iii) Mark (✓) rows (not already marked) which have assignment in marked columns.
(iv) Repeat step (ii) and (iii) until the chain of marking ends.
(v) Draw minimum no. of lines through these marked rows and columns to cover all the zeros.

This procedure will yield the minimum number of lines (equal to the no. of assignments in the maximal assignment obtained in step-3 that will pass through all zeros.

Step-5 Now we select the smallest of the elements that do not have a line through them, subtract it from all the elements that do not have a line passing through them and add it to every element that lies at the # (intersection) of two lines and leave the remaining elements of the matrix unchanged.

Step-6: At the end of step-5, no. of zeros are increased (never decreased) in the matrix than that in step-3 So we repeat step 3 to the modified matrix obtained in step-5 to get the optimal assignment as above ways

Example 1: Solve the following minimal assignment problem:— A department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the task be allocated, one to a man, so as to minimize the total man hours?

Subordinates

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Solution:- Step-1 We first find that whether it is balanced or unbalanced ~~matrix~~ assignment problem. Obviously no. of rows = no. of columns, Hence it is a balanced Assignment problem. So we choose smallest element in each row and subtract them from every ~~other~~ element of the corresponding row. So we get the following matrix:-

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

Step-2 We subtract the smallest element in each column of the row-reduced matrix (obtained in step-1) from every element of the corresponding column, we get the following column reduced matrix. →

Step-3. Now we test whether it is possible to make an assignment using the zeros by the method described in step-3 (Algorithm)

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

Starting with row I, we mark ☐ (i.e, make an assignment) in the row containing only one zero and cross (x) the zeros in the corresponding column in which ☐ lies. Thus, we have the following matrix as under: ~~[illegible]~~.

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

Again Starting with column I, we mark ☐ (i.e, make assignment) in the column containing only one unmarked or uncrossed zero in the above table and cross out the zeros in the corresponding row in which this assignment ☐ is marked. Thus we have the following table:—

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

Let us now see that (i) every row and every column have one assignment or (ii) the no of ☐ = no of order of the matrix.

So we have the complete optimal zeros assignment.

∴

Job	I	II	III	IV
Man	1	3	2	4

which is the optimal assignment.

Assignment Problem (Maximization Problem)

(Spl kind of transportation problem)

Hungerian Method:- Convert it into minimization problem by subtracting all the elements from the largest element. (Same Algo will be applied for Minimization problem).

There are two phases : →

Phase-1:- Reduction by Row and Column.

Phase-2:- Optimizations.

~~Balanced Assignment~~ problem.

Q1: Solve the following assignment problems to maximize sales.

Assignment problem

Each source ~~each~~ should have the capacity to fulfill the demand of any of the .

Teritories.

	I	II	III	IV
A	45	38	30	22
B	35	29	20	14
C	35	29	20	14
D	27	20	15	10

Q. 2. Solve the following Assignment. How the tasks should be allocated to each person so as to minimize the total man hours?

	I	II	III	IV
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

Soln: There are two types of Assignment problems (1) Balanced and (2) Unbalanced. If there is equal no. of rows and columns then it is called balanced otherwise unbalanced Assignment problem. For Unbalanced Assignment problem we will have to add either a dummy rows or dummy colums filling (165) each rows ~~by~~ or Colum by zero element. ⑧

Sol^n: Hungarian Method:-

Step-1 Subtracting the smallest element from each row : Since 8 is the minimum elt So subtracting from each element of the first rows and doing Similarly in each row we get matrix assignment

row reduction

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 0 & 18 & 9 & 3 \\ B & 9 & 24 & 0 & 22 \\ C & 23 & 4 & 3 & 0 \\ D & 9 & 16 & 14 & 0 \\ \hline \end{array}$$

Column reduction : Similarly we will do Column reduction

$$\begin{array}{|cccc|} I & II & III & IV \\ \hline \boxed{0} & 14 & 9 & 13 \\ 9 & 20 & \boxed{0} & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \\ \hline \end{array}$$

← Reduced Matrix.

Here we have to see which Subordinates has to assign what tasks:

Next ~~Now~~ let us assign row wise then Column wise.

If in a row or column if we have single zero i.e, '0' then we will have to make square ($\boxed{0}$)

i.e, we will have to make square & Cross if there is any row having ~~[illegible]~~ single zero. and in that Colum if there is ~~any/all zero~~ then we make a Cross.

i.e, Cross ~~[illegible]~~ to all zeros square means Assignment has been made or completed

In first row we have single zero so we may assign ($\boxed{}$) and ~~in that~~ if in Corresponding ~~If in that this 1st~~ Colum there ~~be~~ will be any or all zero then we had to make cross to all but there is no zero in ~~first~~ corresponding Colum so will not make ~~zero~~ Cross anywhere

Assignment: A-I, B-III, C-II, D-IV total

$\boxed{167}$

Man hours in given matrix 8 4 19 10 = 41 hrs.

Note:- If ~~there is~~ no of Assignment ≠ the order of matrix then we will have to improve the matrix.

Ex-2 For that let us consider the following problem:—

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 18 & 26 & 17 & 11 \\ B & 13 & 28 & 14 & 26 \\ C & 38 & 19 & 18 & 15 \\ D & 19 & 26 & 24 & 10 \\ \hline \end{array}$$

row reduction

$$\begin{array}{|cccc|} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{array}$$

Column Reduction

$$\begin{array}{|cccc|} 7 & 15 & 5 & \boxed{0} \\ \boxed{0} & 11 & \cancel{0} & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{array}$$

Reduced Matrix

We make row wise Assignment:-

Here we have only three Assignment ≠ its order

a in each row/column we have not get at least one assignment.

So optimality is not reached

i.e., Since 4th row and 3rd Column do not have any assignment

So it is not an optimal Assignment.

Therefore we improve the above matrix.

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In second row there is a single zero. ⑨ So we will make an assignment by putting square ([ ]) there. Again There is no any zero in the (corresponding) element of the 2nd rows. So we skip.

In third row we see that there are more than a single zero so we skip.

In the next row we will see that ∃ a single zero. So we assign here and in this particular column there is one zero so we make a cross.

So we have completed row-wise assignment.

Similarly we repeat columnwise assignment.

In the 1st column ∄ any single zero.

In 2nd column; we assign a single zero in this column. We have to check there is any zero in this particular row & zero. There was a zero already ~~ass~~ crossed. Now we find that we had all zero which has been assigned already.

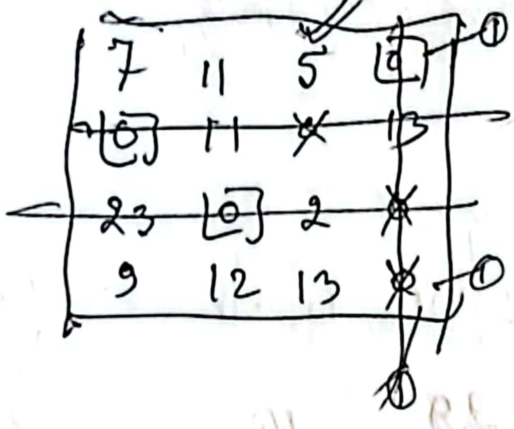
Condition of Optimality: — If ~~No. of rows~~ No. of Assignment (4)

= No. of order of the matrix (4)

Hence optimal Assignment has been made.

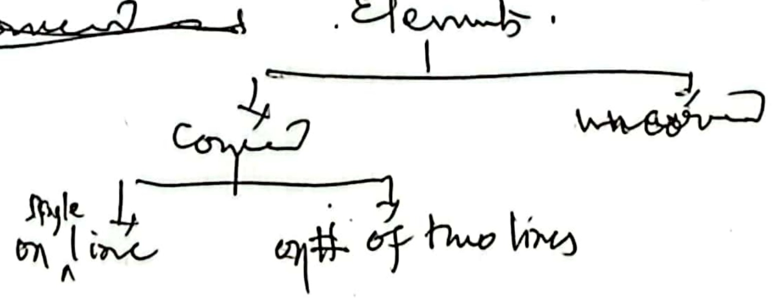
Then check: or In each rows/columns there must be at least one assignment. Then it will be an optimal Assignment.

Let us make the table again &



We have to draw minimum no. of lines on Assigned zeros ~~to that row~~. Let us mark ① that row in which no assignment (x) has been made (4th row) (4th column). To find row in the 4th Colmn ~~which row has~~ is there any zero Assigned earlier. Yes in there is one zero so mark 2nd by the Assigned zero ([0]). Then we have to see unticked row and that column in which we have ticked. We have to draw a line in which row there ~~has~~ is not any tick so 2nd & 3rd row will have a line. Now we see that Colmn in which we have ticked so the fourth Column so that all zeros has been covered. Now after drawing the minimum no. of lines we find for the matrix those elements which has been covered or uncovered (on line)

~~Covered~~



Now we find the smallest element among uncovered elements which is 5.

Now Elements on line will remain same (as it is) and all uncovered element must be subtracted by 5 and add on ~~elements~~ on ++

et
ns
update
this change
table

$$\begin{array}{c|cccc} & I & II & III & IV \\ \hline A & 2 & 6 & \boxed{0} & \cancel{0} \\ B & \boxed{0} & 11 & \cancel{0} & 18 \\ C & 23 & \boxed{0} & 2 & 5 \\ D & 4 & 7 & 8 & \boxed{0} \end{array}$$

(170) Do Assignment Again as done initially (12)

is the optimal Assignment

Optimal Assignment A = III, B – I, C – II, D – IV Totally

~~Ass~~ Man hrs : 30 35 29 10 = 104 hrs

Check in original matrix.

Assignment Problem (71)

Sp. problem of Transportation Problem.

85 (1)

(i) Balanced (ii) Unbalanced
No. of rows = No. of column — convert by adding Dummy row/column.

Hungerian Metho. Algorithm.

Phase-1 Row & Column Reduction.

(1) ~~Solve using the Hungerian.~~

Step-1, Substract the minimum value of each row from the entries of that row.

Step-II Substract the minimum value of each column from the entries of that column.

Phase-2. Optimization of the problem. Algo.

Step-1 — Draw more of lines to cover all the zeros

Step-2. — of the matrix.

Q1. Solve the following assignment problem using Hungarian method. The matrix entries represent the processing times in hours. There are five operators & five jobs. ~~eg~~

Balance Method.

	1	2	3	4	5	Row minimum.
1	9	11	14	11	7	7
2	6	15	13	13	10	6
3	12	13	6	8	8	6
4	11	9	10	12	9	9
5	7	12	14	10		7

If unbalanced then add dummy rows/column having 0 entries.

	1	2	3	4	5
1	2	4	7	4	0
2	0	9	7	7	4
3	6	7	0	2	2
4	2	0	1	3	0
5	0	5	5	3	7
Column Minimum.	0	0	0	2	0

row minimum reduced matrix. after step-I using 2nd step...

New matrix after column minimization.

	1	2	3	4	5
1	2	4	7	2	0
2	0	9	7	5	4
3	6	4	0	0	2
4	2	0	1	1	0
5	0	5	7	1	7

2nd Phase · <u>Optimization of the Problem</u>.

<u>Step-II</u>. Draw minimum no. of lines to cover all the zeros of the matrix.

there are two procedure (a) row scanning
(b) column scanning.

ⓐ <u>row scanning:-</u>

(i) Solving from the first row, ask the following question. Is there exactly one zero in that row? If yes mark a square around that zero in that cell and draw a vertical line passing through that zero; otherwise skip that row.

(ii) After scanning the last row, check whether all the zero's are covered with lines. If yes, go to step-2; otherwise, do column scanning.

ⓑ <u>Column scanning;</u>→

(i) Start from the first column, ask the following question:- Is there exactly one zero in that column? If yes mark a square around that zero entry and draw a <u>horizontal</u> line passing through that zero; otherwise skip that column.

(ii) After scanning the last column check whether all the zeros are covered with lines.

<u>After row & column reduction we get new matrix then.</u>

	1	2	3	4	5
1	2	4	7	2	$\boxed{0}$
2	$\boxed{0}$	9	7	5	4
3	6	7	$\boxed{0}$	0	2
4	2	$\boxed{0}$	①	1	0
5	0	5	7	1	7

← skipp

deleted leaving one zero.

don't have zero.

↑ no zero

After row scanning go for column scanning from 3rd column.

Step 2. Check whether the no. of square marked is equal to the no. of rows of the matrix. If yes, go to step-5, else go to step-3.

Step-3. Identify the minimum value of the undeleted cell values.

(a) add the minimum undeleted cell value at the intersection points of the present matrix.

(b) Subtract the minimum undeleted cell value from all the undeleted cell values.

(c) all other entries remain same.

Step-4. Go to step-1

Step-5. Treat the solution as marked by the square as the optimal solution.

Here no. of squares i.e. 4 ≠ 5 rows. optimality is not reached so go to 3rd step.

6, 7, 2 in 3rd colm.

See step-3.
(a) add the minimum undeleted cell.
(b) Subtract "
(c) ✓

New matrix after step-3. Go for step-4.

	1	2	3	4	5
1	2	4	6	1	$\boxed{0}$
2	$\boxed{0}$	9	6	4	4
3	7	8	$\boxed{0}$	0	3
4	2	$\boxed{0}$	0	0	0
5	0	5	6	$\boxed{0}$	7

step-4. Go to step-1 row & colm scanning.

All the zeroes are deleted.

Check no. of squares = No. of rows.

5 = 5 Hence the solⁿ is optimal & feasible.

Job operator.

Job	1	2	3	4	5
1					
2					
3					
4					
5					

Job	operator.	Time.
1	5	7
2	1	6
3	3	6
4	2	9
	4	7

Transportation Problem:- It is special kind of LPP in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the sources and distinations respectively. Such that the total cost of transportation is minimised.

Types-I Balanced TP S = D
Type-II Unbalanced T.P. S ≠ D

Methods:- (I) Finding the initial basic feasible solution.
(II) Finding optimization.

		A	B	C	supply
Source	I	2	7	5	200
	II	3	4	2	300
	III	5	4	7	500
Demand		200	400	400	1000

← Balance D = S.

I Phase Transportation problem.
(1) Northwest corner cell method.
(2) Least cost cell method
(3) VAM

(I) **N.C.C. Method** Allocate diff of sources & diff of destination.

N.W corner →

	A	B	C	D	Supply
1	250 3	50 1	7	4	(2) 300 − 250 = 50 − 0
2	2	300 6	100 5	9	400 − 300 = 1000
3	8	3	300 3	200 2	500 − 300 = 200 0
	250 0	350 −50 = 300 0	400 −100 = 300 0	200 0	1200 S = D Balance.

Multiply as follow

$$(250 \times 3) + (50 \times 1) + (300 \times 6) + (100 \times 5) + (300 \times 3) + (200 \times 2)$$

= Rs 4400 Ans

II Least Cost Cell method. (176)

Least Cost among the various cell. (Q)

Destination

	A	B	C	D	Supply
1	3	300 1	7	4	~~300~~ 0
Source 2	250 2	50 6 * D=3	100 5	9	~~400~~ ~~150~~ ~~50~~ 0
3	8	3	300 3	200 2	~~400~~ ~~500~~ ~~300~~ 0
demand	~~250~~	~~350~~	~~400~~	~~200~~	
	0	~~50~~ 0	~~100~~ 0	0	

(300×1) + (250×2) + (50×6) + (100×5) + (300×3) + (200×2) = 2900 //

III Vogel's Approximation method.

① we need to identify Row diff. for each column & row

② select max^m penality occurs in row II

	A	B	C	D	Supply	Row Diff.			
I	3	300 1	7	4	~~300~~ 0	2	③	—	—
II	250 2	6	150 D=3 5	9	~~400~~ (150)	③	1	1	1
III	8	50 3	250 3	200 2	~~500~~ ~~300~~ ~~250~~ 0	1	1	1	0
Demand	~~250~~ 0	~~350~~ ~~50~~ 0	~~400~~ (150)	~~200~~ 0					
Column Diff.	1	2	2	2					
		2	2	2					
		3	2	⑦					
		③	2						

Total Tr. Cost with the allocated cell = (300×1) + (250×2) + (150×5) + (50×3) + (250×3) + (200×2) = 2850 //